

# A new solution to the Boltzmann equation and its hydrodynamical limit

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Collaborators: *G. Denicol, U. Heinz, J. Noronha and M. Strickland*  
Based on: **PRL 113 202301 (2014), PRD 90 125026 (2014)**

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**THE OHIO STATE UNIVERSITY**

# Outline

- Motivation: Success of viscous hydrodynamics
- A short overview of the relativistic Boltzmann equation
  - Symmetries in action: G. Baym's solution
- Exact solution to the Boltzmann equation undergoing Gubser flow
- Testing the validity of different hydrodynamical approximations
- Conclusions and outlook

# Success of viscous hydrodynamics

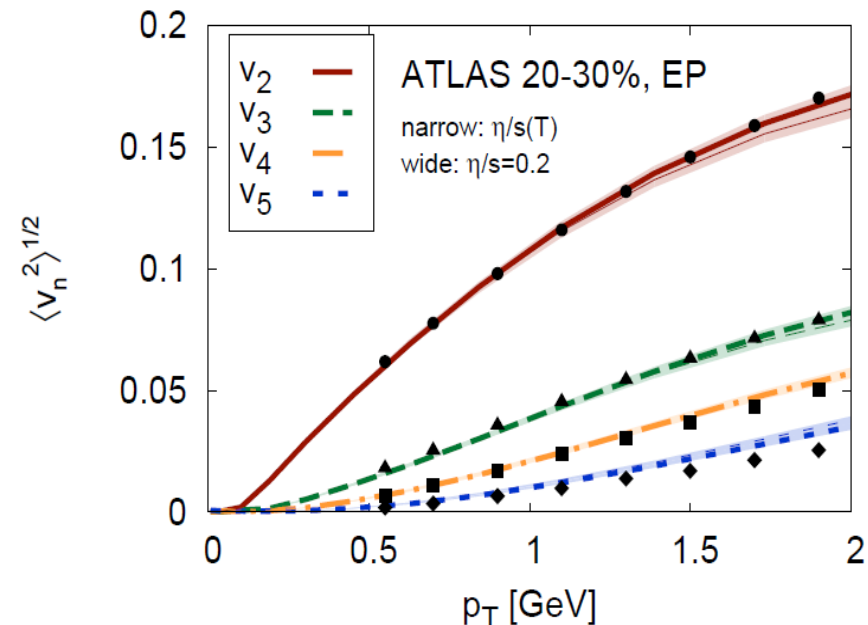
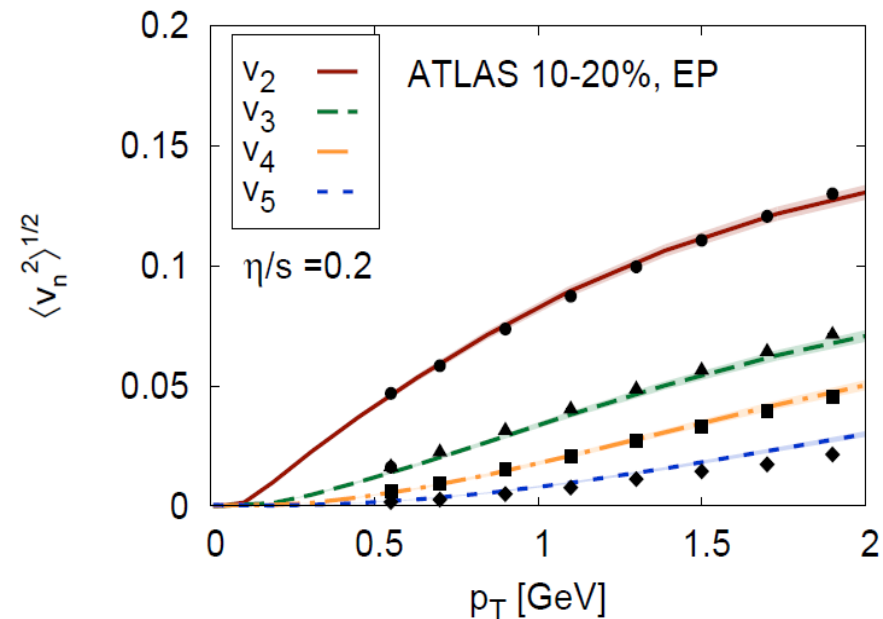
**Quark gluon plasma: the hottest, tiniest and most perfect fluid ever made on Earth:**

$$\frac{\eta}{s} = \frac{2}{4\pi} \pm 50\%$$

**Hydro requires as an input:**

1. **Initial conditions:** CGC, Glauber, etc.
2. **Evolution for the dissipative fields:** 2<sup>nd</sup> order viscous hydro
3. **EOS:** lattice + hadron resonance gas
4. **Hadronization and afterburning** URQMD, etc.

**What is the best hydrodynamical description that describes the QGP?**

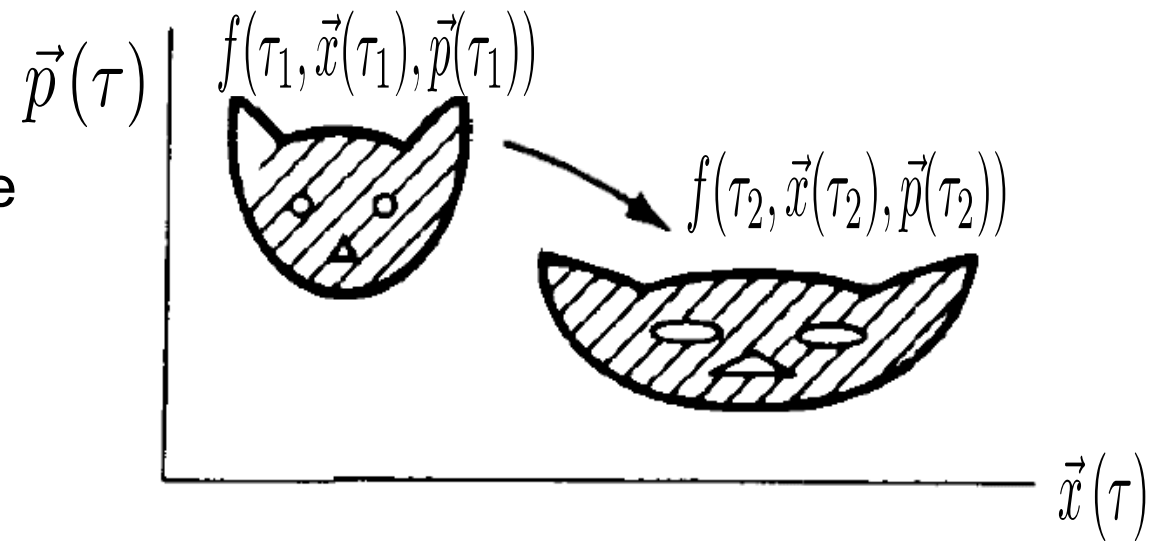


Gale et. al, PRL 110, 012302 (2012)

# A short overview of the relativistic Boltzmann equation

The distribution function is defined in a 7-dimensional phase space

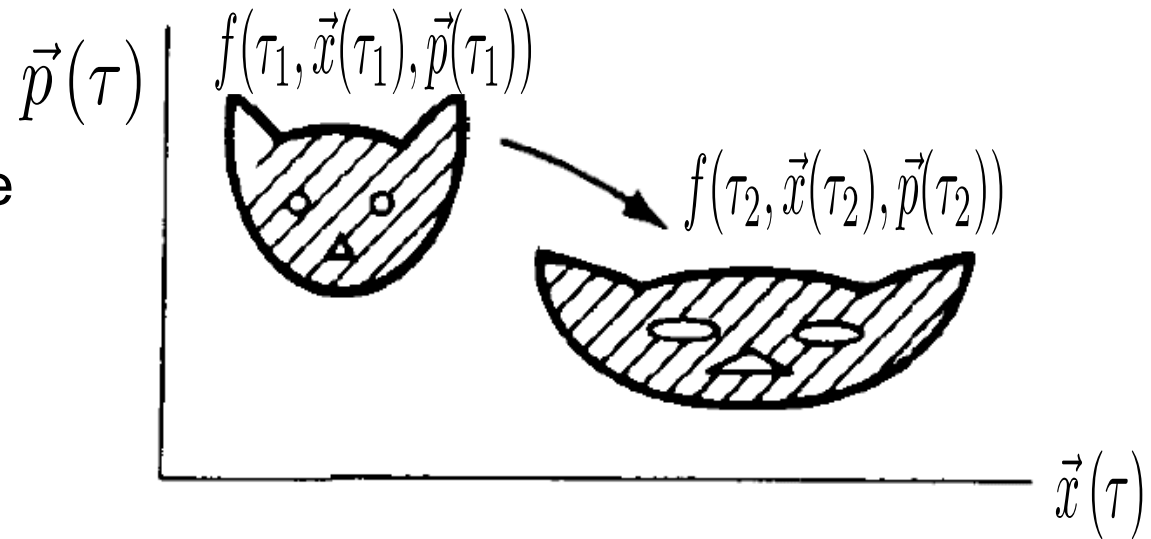
$$f = f(x^\mu, p_i)$$



# A short overview of the relativistic Boltzmann equation

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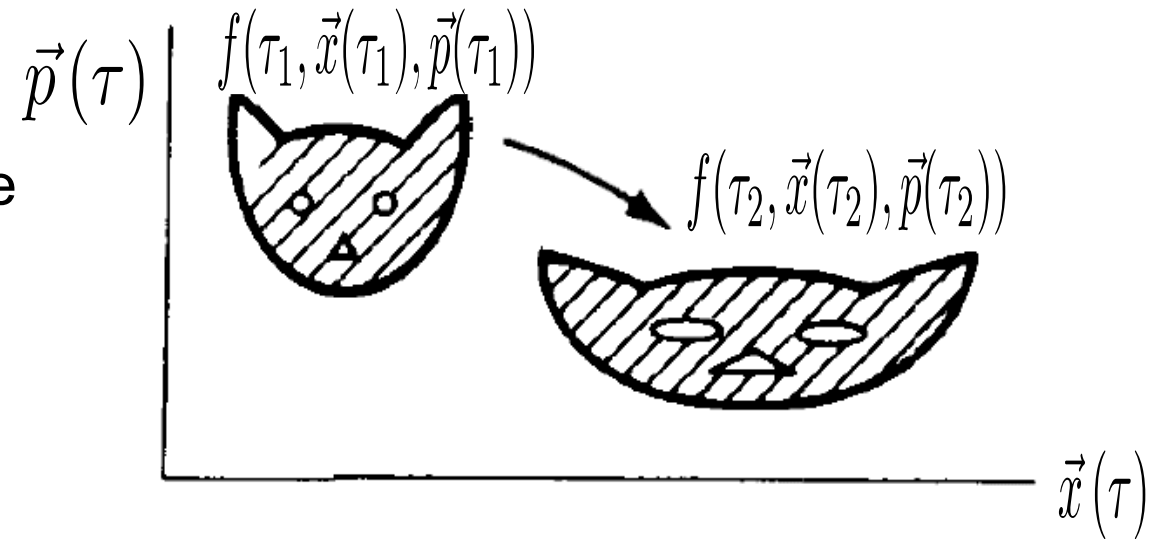
For on-shell particles, the evolution of the distribution function is determined by the Boltzmann equation

$$\frac{df}{d\tau} = \frac{dx^\mu}{d\tau} \frac{\partial f}{\partial x^\mu} + \frac{dp^i}{d\tau} \frac{\partial f}{\partial p^i} = \mathcal{C}[f]$$

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For on-shell particles, the evolution of the distribution function is determined by the Boltzmann equation

$$\frac{df}{d\tau} = \underbrace{\frac{dx^\mu}{d\tau} \frac{\partial f}{\partial x^\mu}}_{\text{Diffusion}} + \underbrace{\frac{dp^i}{d\tau} \frac{\partial f}{\partial p^i}}_{\text{External Force}} = \underbrace{\mathcal{C}[f]}_{\text{Collisional kernel}}$$

# A short overview of the relativistic Boltzmann equation

In a general curvilinear system and in the absence of external fields, the Boltzmann equation is written as

$$p^\mu \frac{\partial f}{\partial x^\mu} + \Gamma_{\mu i}^\lambda p_\lambda p^\mu \frac{\partial f}{\partial p_i} = \mathcal{C}[f]$$

Given a distribution function, one can calculate the macroscopic quantities by computing its moments

$$\varepsilon(x) = \int \frac{d^3 p}{\sqrt{-g} p^0} (p \cdot u)^2 f(x^\mu, p_i) ,$$

$$\mathcal{P}(x) = \frac{1}{3} \int \frac{d^3 p}{\sqrt{-g} p^0} \Delta_{\mu\nu} p^\nu p^\mu f(x^\mu, p_i) ,$$

$$\pi^{\mu\nu}(x) = \int \frac{d^3 p}{\sqrt{-g} p^0} p^{\langle\mu} p^{\nu\rangle} f(x^\mu, p_i) .$$

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In a general curvilinear system and in the absence of external fields, the Boltzmann equation is written as

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In general, collisional kernel is extremely difficult to calculate (e.g. AMY JHEP 0305 (2003) 051).

In this work we will use a simplified collisional kernel in the **relaxation time approximation (RTA)**

$$\mathcal{C}[f] = \frac{(p \cdot u)}{\tau_{rel}} [f(x^\mu, p_i) - f_{eq}(x^\mu, p_i)]$$

$\tau_{rel}$  : relaxation time

$f_{eq} = e^{u \cdot p / T}$  : equilibrium distribution function

# Symmetries in action: G. Baym's solution

- G. Baym (1984) solved exactly the Boltzmann equation within the RTA approximation for a system undergoing Bjorken flow
- An elegant way to obtain this solution is by considering the constraints over the distribution function due to the symmetries associated to the Bjorken flow

$$ISO(2) \otimes SO(1,1) \otimes Z_2$$

$$Z_2 \longrightarrow \text{Reflections along the beam line} \quad z \rightarrow -z$$

$$SO(1,1) \longrightarrow \text{Longitudinal Boost invariance} \quad \xi_1 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}$$

$$ISO(2) \longrightarrow \text{Translations in the transverse plane + rotation along the longitudinal } z \text{ direction}$$

$$\begin{aligned} \xi_2 &= \frac{\partial}{\partial x} & , \xi_3 &= \frac{\partial}{\partial y} \\ \xi_4 &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \end{aligned}$$

# Symmetries in action: G. Baym's solution

- G. Baym (1984) solved exactly the Boltzmann equation within the RTA approximation for a system undergoing Bjorken flow
- An elegant way to obtain this solution is by considering the constraints over the distribution function of the isometry group associated to the Bjorken flow

$$ISO(2) \otimes SO(1,1) \otimes Z_2$$

- Bjorken flow is constructed by finding a time-like vector which is invariant under this isometry group, i.e.,

$$[\xi_i, u^\mu] = 0 \quad \longrightarrow \quad u^\mu = \frac{1}{\sqrt{t^2 - z^2}} (t, 0, 0, z)$$

- In Milne coordinates  $(t, x, y, z) \rightarrow (\tau, x, y, \varsigma)$   $\tau = \sqrt{t^2 - z^2}$   
 $\tanh \varsigma = z/t$

$$u^\mu = (1, 0, 0, 0) \quad \longrightarrow \quad \text{Static flow}$$

# Symmetries in action: G. Baym's solution

- ISO(2) symmetry implies:
  - No dependence on x and y
  - Depends on the total transverse momentum:

$$p_{\perp} = \sqrt{p_x^2 + p_y^2}$$

- SO(1,1) symmetry implies:
  - Depends on the longitudinal proper time

$$\tau = \sqrt{t^2 - z^2}$$

- Depends on the following combination of variables

$$\omega = tp_z - zE$$

- Z<sub>2</sub> symmetry implies:

$$z \rightarrow -z \Rightarrow \omega \rightarrow -\omega$$

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- $Z_2$  symmetry implies:

$$z \rightarrow -z \Rightarrow \omega \rightarrow -\omega$$

$$f(\underbrace{t, x, y, z, p_x, p_y, p_z}_{\text{7 dimensional phase space}})$$

7 dimensional  
phase space

Milne coordinates



$$f(\tau, p_{\perp}, \omega)$$

$f$  depends **only** on three independent variables!!!

# Symmetries in action: G. Baym's solution

In principle, the Boltzmann equation is written as

$$p^t \partial_t f + p_x \partial_x f + p_y \partial_y f + p^z \partial_z f = \frac{p \cdot u}{\tau_{rel}} (f - f_{eq})$$

Constraints of the isometry group (Bjorken flow)+ change to Milne coordinates

$$p^t = \sqrt{m^2 + p_x^2 + p_y^2 + p_z^2}$$

$$\partial_\tau f = -\frac{1}{\tau_{rel}(\tau)} (f - f_{eq})$$

This equation can be solved exactly (Baym, PLB 138 (1984) 18)

$$f(\tau, p_\perp, p_\varsigma) = D(\tau, \tau_0) f_0(\tau_0, p_\perp, p_\varsigma) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{rel}(\tau')} D(\tau', \tau_0) f_{eq}(\tau', p_\perp, p_\varsigma)$$

$$D(\tau, \tau_0) = \exp - \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{rel}(\tau')}$$

Generalization to highly anisotropic systems:  
Strickland, Ryblewski, Florkowski, Maksymiuk.

# Lessons learned from G. Baym's solution

- Symmetries imposes strict **constraints** on the functional dependence of the distribution function
  1. Number of independent variables
  2. Particular combination in which dependent variables appear
- It is important to choose the **correct** coordinate system where the symmetries are explicitly manifest
  - Bjorken expanding system is homogeneous in Milne coordinates.
  - Bjorken flow velocity is invariant under the Bjorken isometry group. In the Milne coordinates it is a static flow.
- Bjorken's flow **does not include** transverse expansion due to translational invariance in the transverse plane.
- Is there an analytical way to go beyond the 0+1 dim Bjorken flow?
  - ➡ Yes, **Gubser flow**. (S. Gubser PRD 82 (2010) 085027)
- This solution has been used as a toy model for very central collisions at LHC to study the hydrodynamic response to small azimuthal asymmetries (Shuryak, Romatschke, Hatta, Noronha, Xiao....)

# Symmetries of the Gubser flow

$$SO(3)_q \otimes SO(1, 1) \otimes Z_2$$

In the Milne (polar) coordinates  $x^\mu = (\tau, r, \phi, \varsigma)$

$$Z_2$$

$$z \rightarrow -z$$

Reflections along the  
beam line

$$SO(1, 1)$$

$$\xi_1 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}$$

Boost invariance

$$SO(3)_q$$

$$\xi_i = \frac{\partial}{\partial x^i} + q^2 \left( 2x^i x^\mu \frac{\partial}{\partial x^\mu} - x^\mu x_\mu \frac{\partial}{\partial x^i} \right) \quad i = 2, 3$$

$$\xi_4 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

Special Conformal  
transformations +  
rotation along the  
beam line

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$$q \rightarrow 0$$

$$SO(3)_q \rightarrow ISO(2)$$

Bjorken's flow is  
recovered

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**SO(3) is associated  
with rotations**

**What are we  
rotating?**

# Weyl rescaling + Coordinate transformation

The metric in Milne (polar) coordinates  $x^\mu = (\tau, r, \phi, \varsigma)$

$$ds^2 = -d\tau^2 + \tau^2 d\varsigma^2 + dr^2 + r^2 d\phi^2$$

**Weyl rescaling**

$$d\hat{s}^2 = \frac{ds^2}{\tau^2}$$

**Coordinate transformation**

$$\rho = -\sinh^{-1} \left( \frac{1 - q^2 \tau^2 + q^2 r^2}{2q\tau} \right)$$

$$\theta = \tanh^{-1} \left( \frac{2qr}{1 + q^2 \tau^2 - q^2 r^2} \right)$$

$$d\hat{s}^2 = -d\rho^2 + \cosh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) + d\varsigma^2$$

***Metric in  $dS_3 \times R$  space!!***

# Gubser's flow velocity profile

Symmetries in this case are better understood after a Weyl rescaling + Coordinate transformation

In the de Sitter space, the generators of  $SO(3)_q$  are

$$\xi_2 = 2q \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$\xi_3 = 2q \left( \cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$\xi_4 = \frac{\partial}{\partial \phi}$$

$SO(3)$  symmetry is **manifest** and it corresponds to rotations in the  $(\theta, \phi)$  subspace.

- So the only invariant flow compatible with the symmetries is

$$[\xi_i, \hat{u}] = 0 \Rightarrow \hat{u}^\mu = (1, 0, 0, 0) \longrightarrow \textbf{Static flow in de Sitter space}$$

# Gubser's flow velocity profile

The flow velocity in Minkowski space is easily calculated:

$$u_\mu = \tau \frac{\partial \hat{x}^\nu}{\partial x^\mu} \hat{u}_\nu \quad \longrightarrow \quad u^\mu = (\cosh \kappa(\tau, r), \sinh \kappa(\tau, r), 0, 0)$$

$$\kappa(\tau, r) = \tanh^{-1} \left( \frac{2q^2 \tau r}{1 + q^2 \tau^2 + q^2 r^2} \right)$$

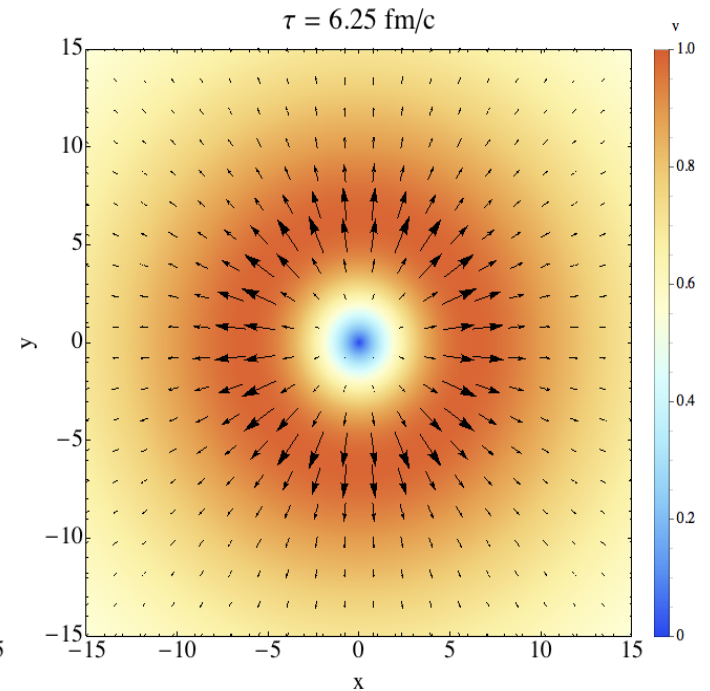
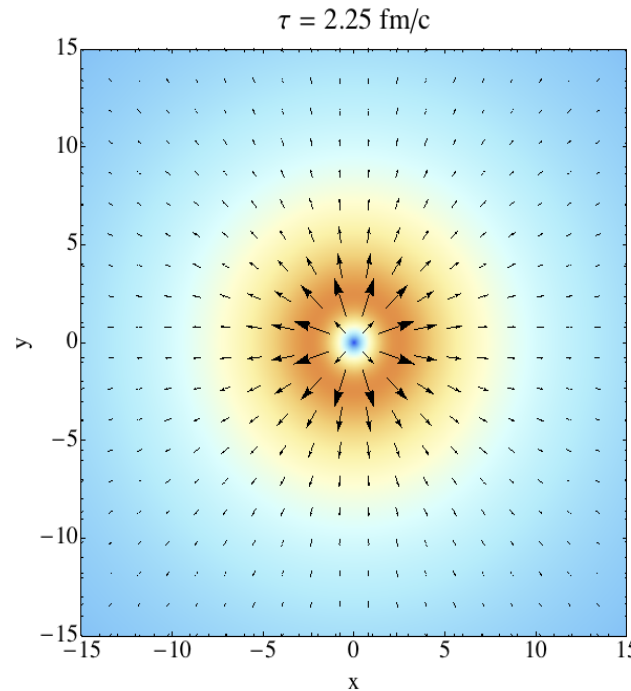
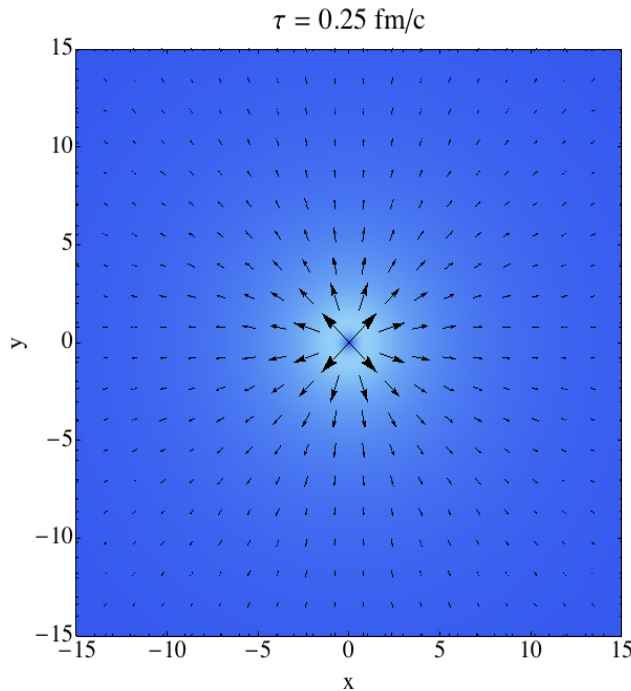
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**Non trivial  
radial flow**

$$\kappa(\tau, r) = \tanh^{-1} \left( \frac{2q^2 \tau r}{1 + q^2 \tau^2 + q^2 r^2} \right)$$



# Exact solution to the RTA Boltzmann equation

**We construct a solution which is invariant under the group  $SO(3)_q \otimes SO(1, 1) \otimes Z_2$   work in the **de Sitter** space**

- **In principle**  $f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$

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$$\hat{p}_\varsigma \sim \omega$$

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$$SO(3)_q \longrightarrow f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \cancel{\theta}, \cancel{\phi}, \varsigma, \underbrace{\hat{p}_\theta, \hat{p}_\phi}_{\hat{p}_\Omega^2}, \hat{p}_\varsigma)$$

$$\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2 \theta}$$

# Exact solution to the RTA Boltzmann equation

Thus the symmetries of the Gubser flow imply

$$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$



$$SO(3)_q \otimes SO(1, 1) \otimes Z_2$$

$$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma)$$

The RTA Boltzmann equation gets reduced to

$$\frac{\partial}{\partial \rho} f(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma) = -\frac{1}{\hat{\tau}_{rel}} \left( f(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma) - f_{eq}(\hat{p}^\rho / \hat{T}(\rho)) \right)$$

Due to **Weyl invariance**  $\hat{\tau}_{rel} = c / \hat{T}(\rho)$

**c** is related with the shear viscosity over entropy ratio (Florkowski et. al, PRC88 (2013) 024903)

$$c = 5 \frac{\eta}{S} \iff \frac{\eta}{S} = \frac{1}{5} \hat{\tau}_{rel} \hat{T}$$

# Exact solution to the RTA Boltzmann equation

This RTA Boltzmann equation in de Sitter space is solved **exactly** and its solution is

$$f(\rho, \hat{p}_\Omega^2, \hat{p}_\xi) = D(\rho, \rho_0) f_0(\rho_0, \hat{p}_\Omega^2, \hat{p}_\xi) + \frac{1}{c} \int_{\rho_0}^{\rho} d\rho' D(\rho, \rho') \hat{T}(\rho') f_{eq}(\rho', \hat{p}_\Omega^2, \hat{p}_\xi)$$

$$D(\rho, \rho_0) = \exp \left\{ - \int_{\rho_0}^{\rho} d\rho' \frac{\hat{T}(\rho')}{c} \right\} \quad f_0 = f_{eq} = e^{\hat{u} \cdot \hat{p} / \hat{T}}$$

# Exact solution to the RTA Boltzmann equation

This RTA Boltzmann equation in de Sitter space is solved **exactly** and its solution is

$$f(\rho, \hat{p}_\Omega^2, \hat{p}_\zeta) = D(\rho, \rho_0) f_0(\rho_0, \hat{p}_\Omega^2, \hat{p}_\zeta) + \frac{1}{c} \int_{\rho_0}^{\rho} d\rho' D(\rho, \rho') \hat{T}(\rho') f_{eq}(\rho', \hat{p}_\Omega^2, \hat{p}_\zeta)$$

$$D(\rho, \rho_0) = \exp \left\{ - \int_{\rho_0}^{\rho} d\rho' \frac{\hat{T}(\rho')}{c} \right\} \quad f_0 = f_{eq} = e^{\hat{u} \cdot \hat{p} / \hat{T}}$$

Giving this exact solution we calculate the energy density and the shear viscous tensor **EXACTLY**

$$\hat{\varepsilon}(\rho) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d\hat{p}_\zeta \int_{-\infty}^{\infty} \frac{d\hat{p}_\theta}{\cosh \rho} \int_{-\infty}^{\infty} \frac{d\hat{p}_\phi}{\cosh \rho \sin \theta} \frac{1}{\hat{p}^\rho} (\hat{p}^\rho)^2 f(\rho, \hat{p}_\Omega^2, \hat{p}_\zeta)$$

$$\hat{\pi}^{\mu\nu} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d\hat{p}_\zeta \int_{-\infty}^{\infty} \frac{d\hat{p}_\theta}{\cosh \rho} \int_{-\infty}^{\infty} \frac{d\hat{p}_\phi}{\cosh \rho \sin \theta} \frac{1}{\hat{p}^\rho} \hat{p}^{\langle\mu} \hat{p}^{\nu\rangle} f(\rho, \hat{p}_\Omega^2, \hat{p}_\zeta)$$

# Exact solution to the RTA Boltzmann equation

This RTA Boltzmann equation in de Sitter space is solved **exactly** and its solution is


$$f(\rho, \hat{p}_\Omega^2, \hat{p}_\xi) = D(\rho, \rho_0) f_0(\rho_0, \hat{p}_\Omega^2, \hat{p}_\xi) + \frac{1}{c} \int_{\rho_0}^{\rho} d\rho' D(\rho, \rho') \hat{T}(\rho') f_{eq}(\rho', \hat{p}_\Omega^2, \hat{p}_\xi)$$

$$D(\rho, \rho_0) = \exp \left\{ - \int_{\rho_0}^{\rho} d\rho' \frac{\hat{T}(\rho')}{c} \right\} \quad f_0 = f_{eq} = e^{\hat{u} \cdot \hat{p} / \hat{T}}$$

The energy-momentum conservation  $\nabla_\mu T^{\mu\nu} = 0$  implies

$$\hat{\varepsilon}_{eq}(\rho) = \hat{\varepsilon}(\rho)$$

**Landau matching condition**


$$\hat{T}^4(\rho) = D(\rho, \rho_0) \mathcal{H} \left( \frac{\cosh \rho_0}{\cosh \rho} \right) \hat{T}^4(\rho_0) + \frac{1}{c} \int_{\rho_0}^{\rho} d\rho' D(\rho, \rho') \mathcal{H} \left( \frac{\cosh \rho'}{\cosh \rho} \right) \hat{T}^5(\rho')$$

$$\mathcal{H}(x) = \frac{1}{2} \left\{ x^2 + x^4 \frac{\tanh^{-1}(\sqrt{1-x^2})}{\sqrt{1-x^2}} \right\}$$

# Conformal hydrodynamic theories in $dS_3 \otimes R$

## Energy momentum conservation

$$\frac{1}{\hat{T}} \frac{d\hat{T}}{d\rho} + \frac{2}{3} \tanh \rho = \frac{1}{3} \bar{\pi}_\zeta^\zeta \tanh \rho$$

## 2<sup>nd</sup>. Order viscous hydrodynamics

**Israel-Stewart (IS)**  $\longrightarrow \partial_\rho \bar{\pi}_\zeta^\zeta + \frac{\bar{\pi}_\zeta^\zeta}{\hat{\tau}_\pi} \tanh \rho + \frac{4}{3} (\bar{\pi}_\zeta^\zeta)^2 = \frac{4}{15} \tanh \rho$

**Denicol et. al. (DNMR)**  $\longrightarrow \partial_\rho \bar{\pi}_\zeta^\zeta + \frac{\bar{\pi}_\zeta^\zeta}{\hat{\tau}_\pi} \tanh \rho + \frac{4}{3} (\bar{\pi}_\zeta^\zeta)^2 = \frac{4}{15} \tanh \rho + \frac{10}{7} \bar{\pi}_\zeta^\zeta \tanh \rho$

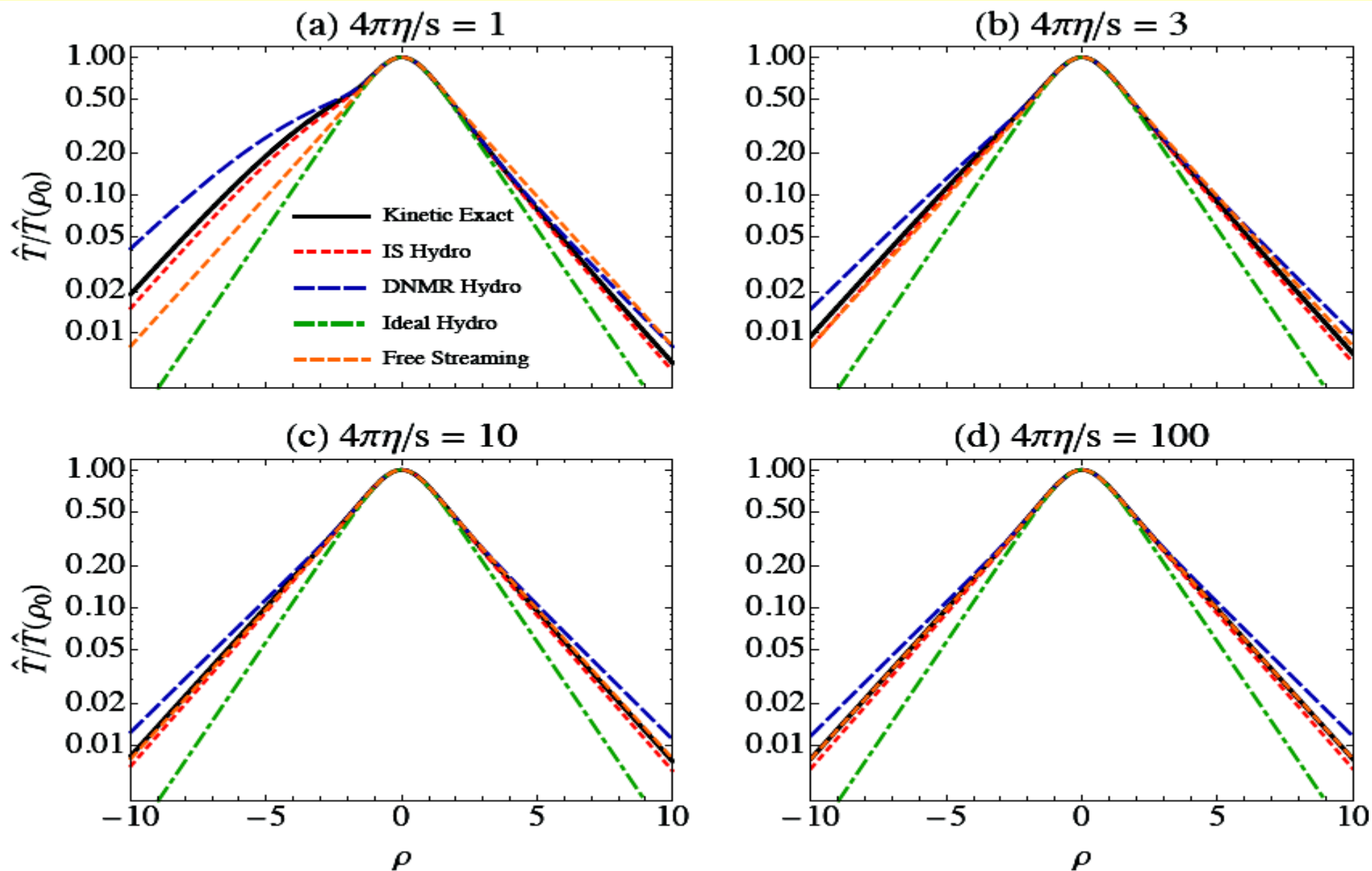
$$\hat{\tau}_\pi = 5\eta/(\hat{S}\hat{T})$$

$$\bar{\pi}_\zeta^\zeta \equiv \pi_\zeta^\zeta / (\hat{T}\hat{S})$$

In this work we also consider two interesting limits:

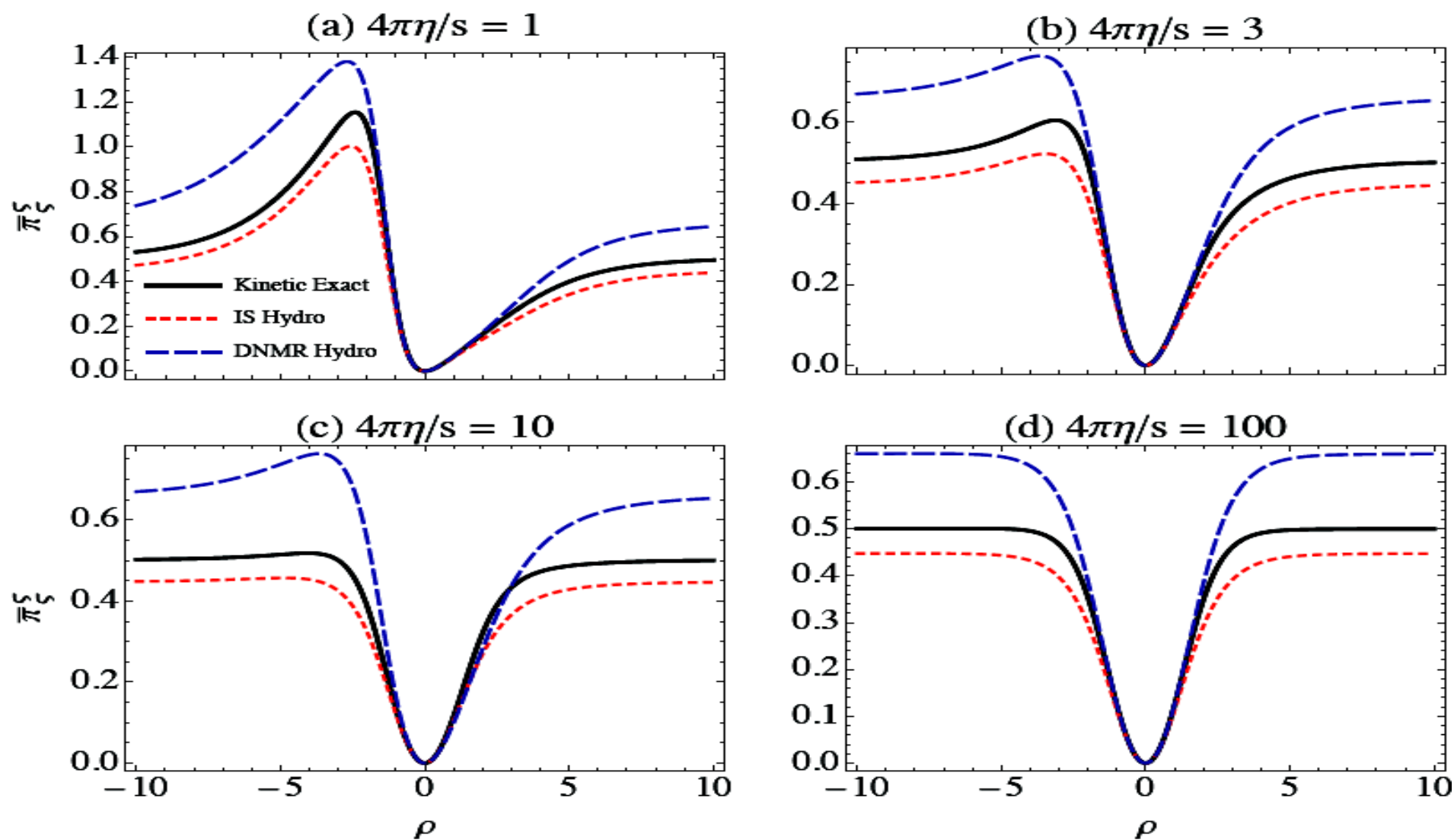
- Free streaming  $\eta/s \rightarrow \infty$
- Ideal hydrodynamics  $\eta/s \rightarrow 0$

# Comparison in $dS_3 \otimes \mathbb{R}$ : Temperature



$$\rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$$

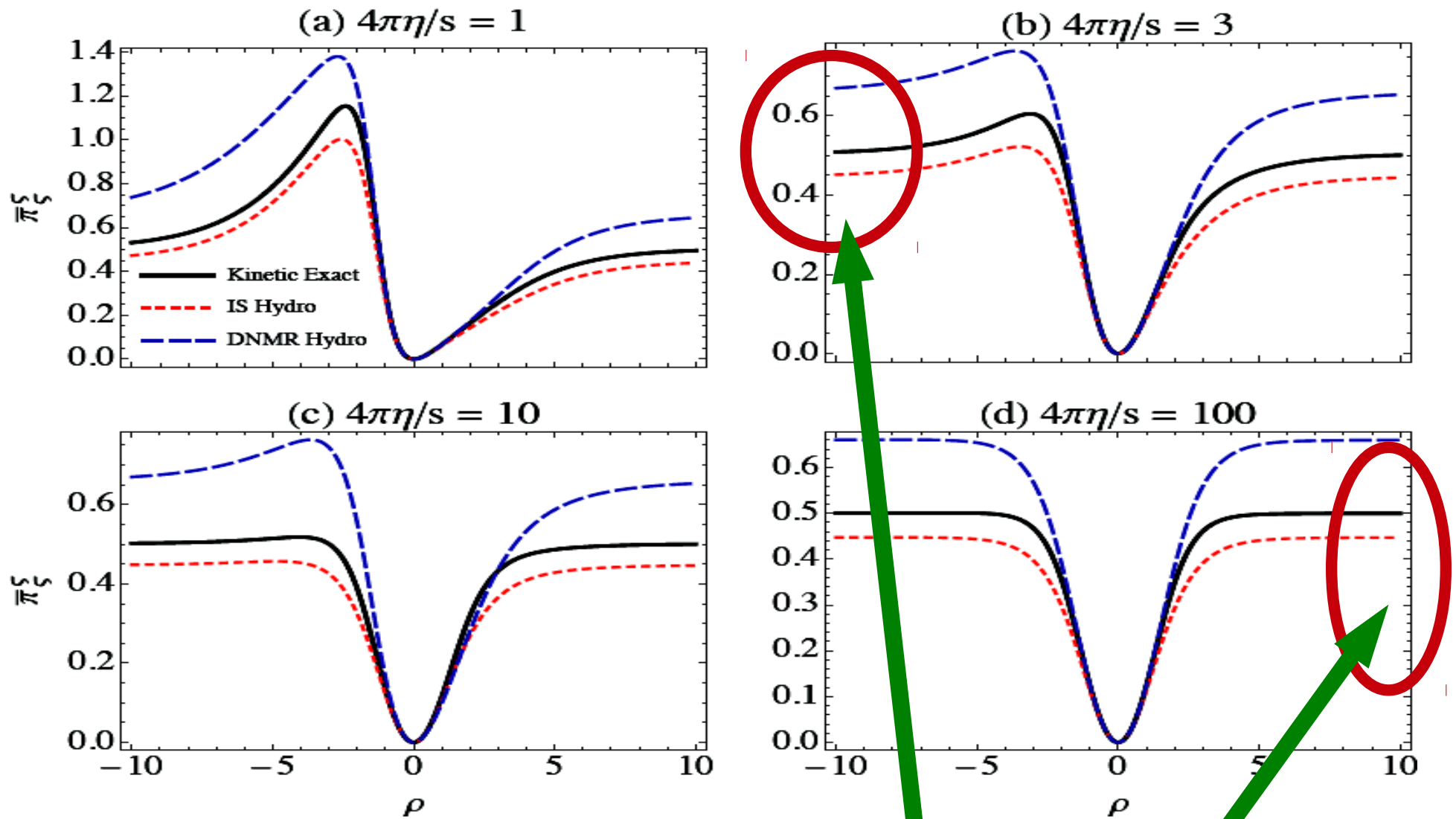
# Comparison in $dS_3 \otimes \mathbb{R}$ : Shear viscous



$$\bar{\pi}_\zeta^\zeta \equiv \pi_\zeta^\zeta / (\hat{T} \hat{S})$$

$$\rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$$

# Comparison in $dS_3 \otimes R$ : Shear viscous



$$\bar{\pi}_\zeta^\zeta \equiv \pi_\zeta^\zeta / (\hat{T} \hat{S})$$

$$\rho_0 = 0 \quad \hat{E}(\rho_0) = 1$$

$\bar{\pi}_\zeta^\zeta$  is large when  $\rho \rightarrow \pm \infty$   
 Large anisotropies

# Knudsen number in $dS_3 \otimes \mathbb{R}$

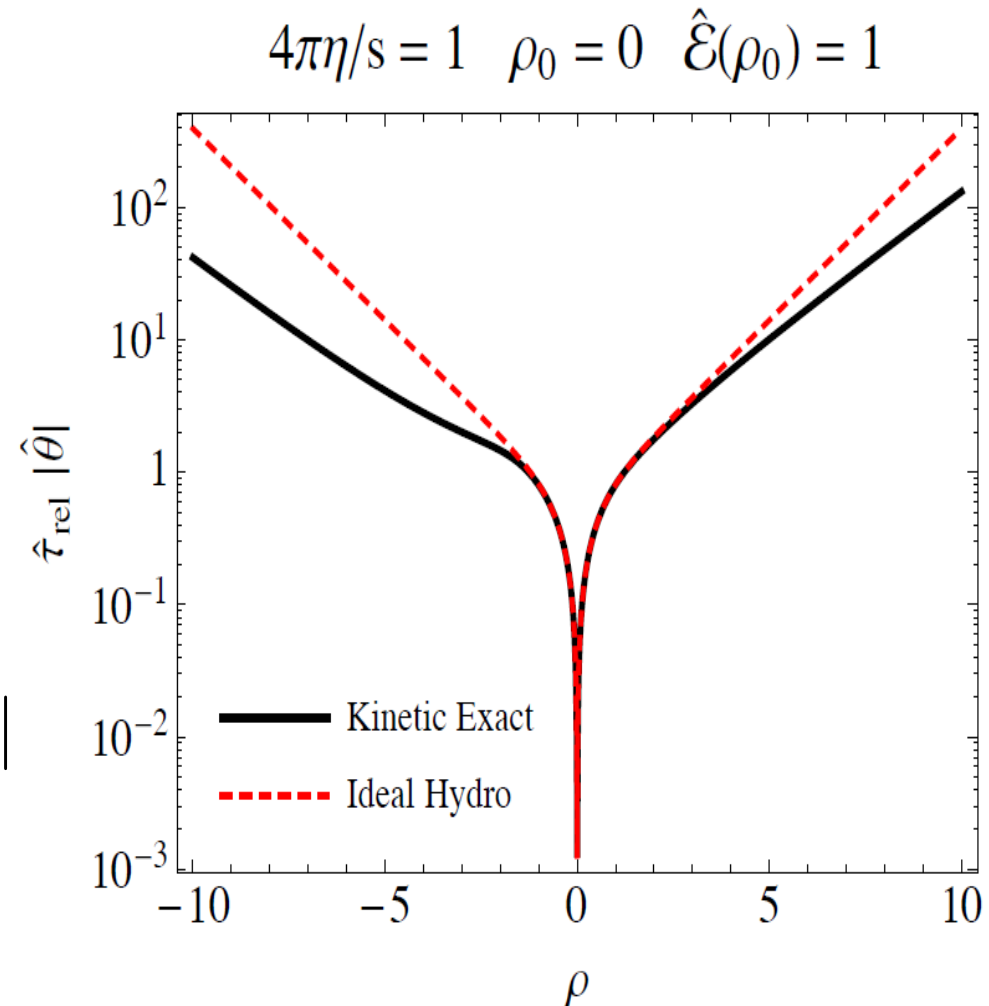
Deviations between 2<sup>nd</sup>. Order viscous hydro and the exact solution are  $\sim 30\%$ . **Why?**

$$\begin{aligned} \text{Kn} &= \hat{\tau}_{rel} |\hat{\nabla} \cdot \hat{u}| \\ &= 2c \frac{\tanh \rho}{\hat{T}(\rho)} \end{aligned}$$

**Ideal Hydro:**

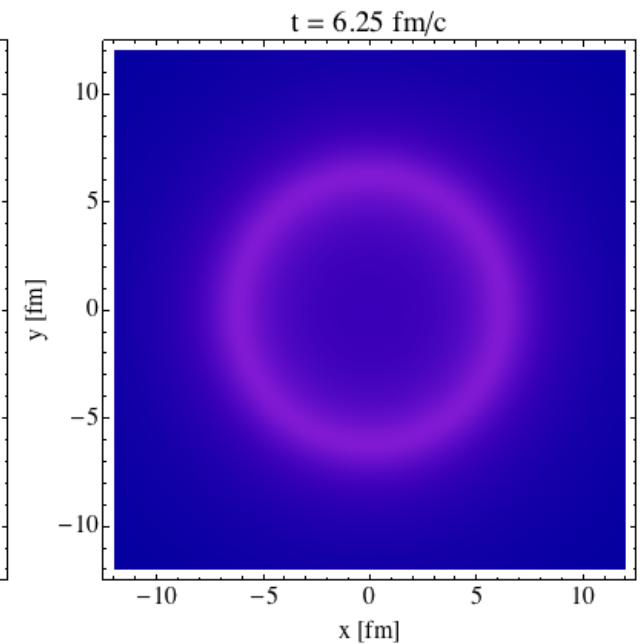
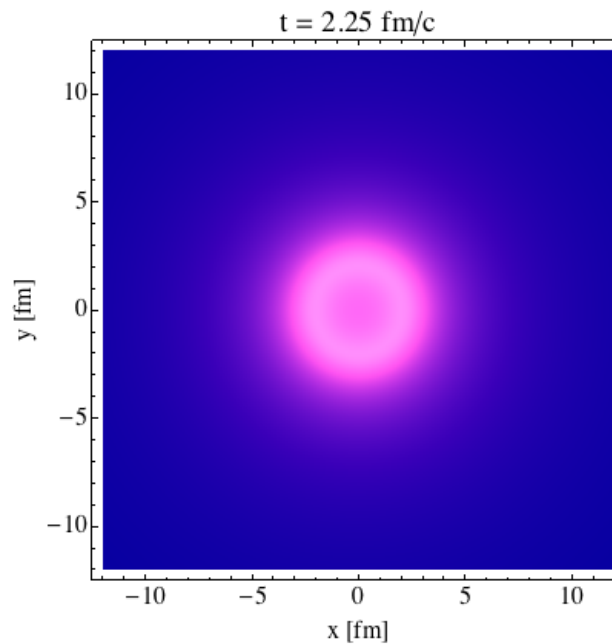
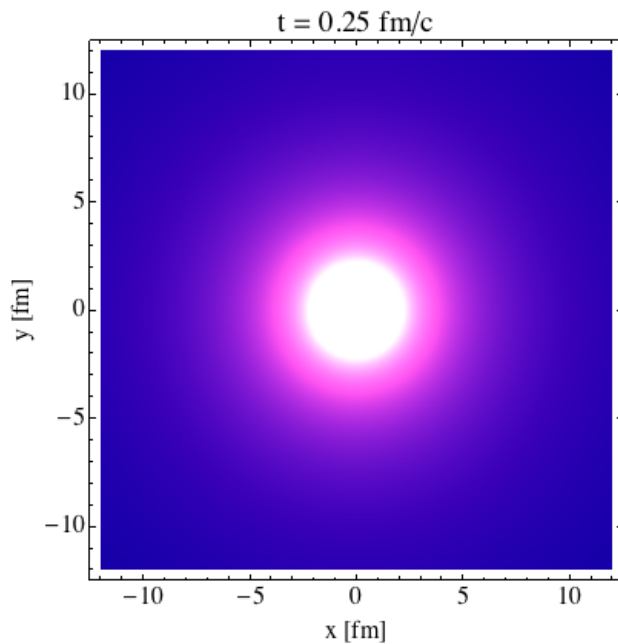
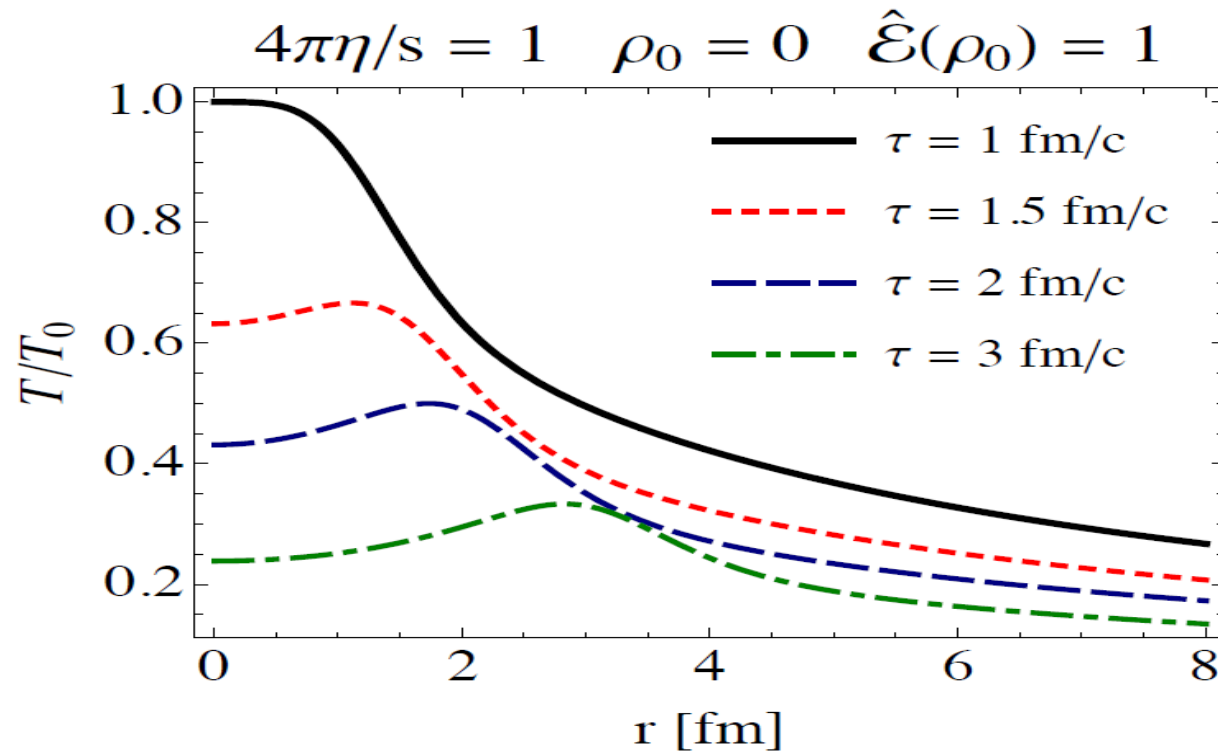
$$\text{Kn}_{ideal} = 2 \frac{c}{\hat{T}_0} |\tanh^{1/3}(\rho) \sinh^{2/3}(\rho)|$$

$$\lim_{\rho \pm \infty} \text{Kn}_{ideal} \sim e^\rho$$

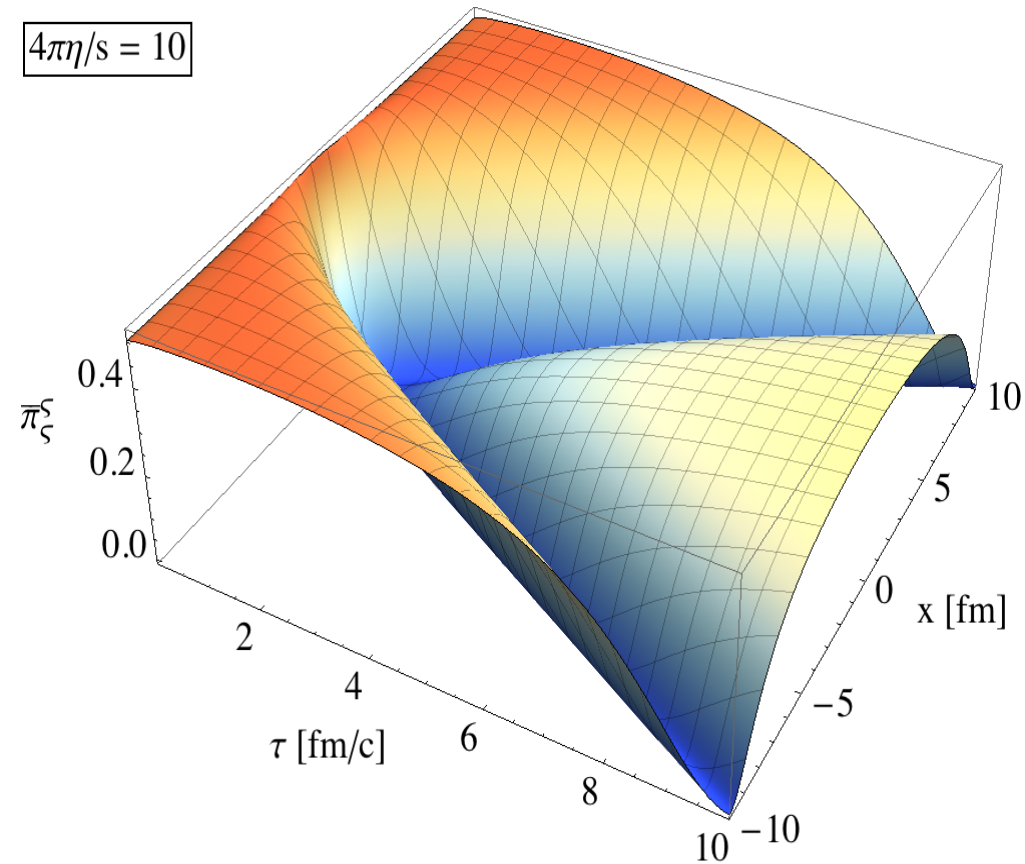
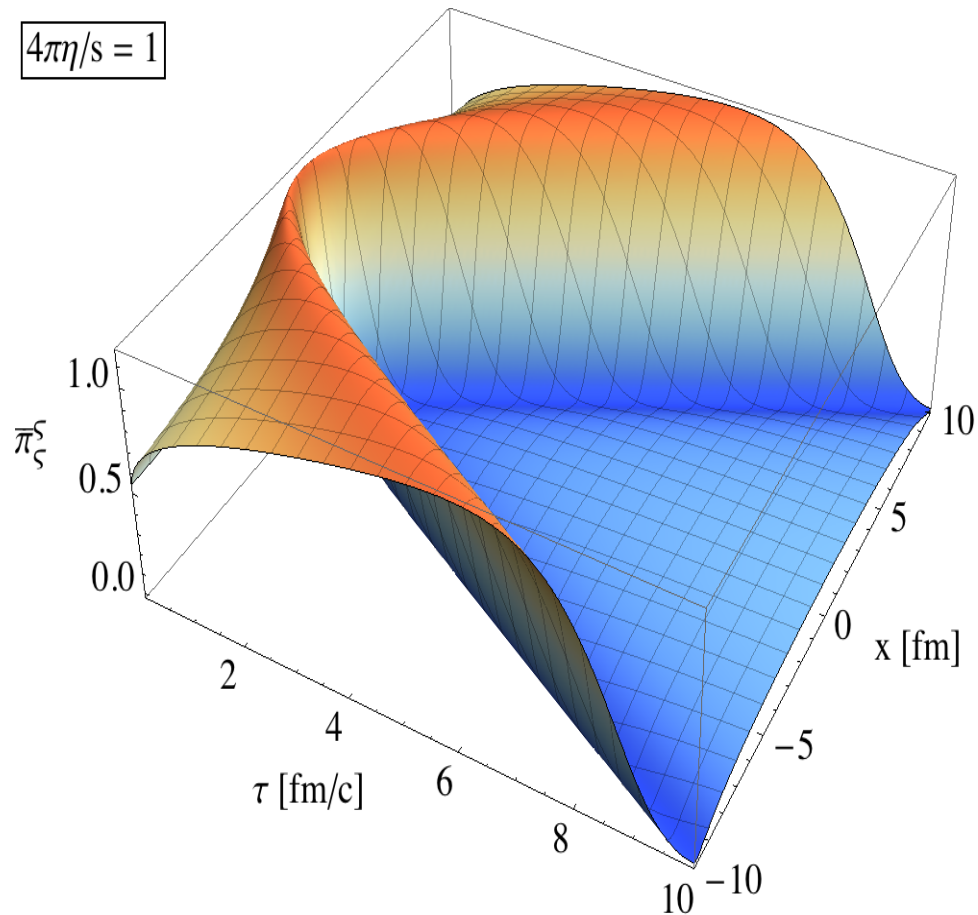


# Temperature in Minkowski space

$$T(\tau, r) = \frac{\hat{T}(\rho(\tau, r))}{\tau}$$

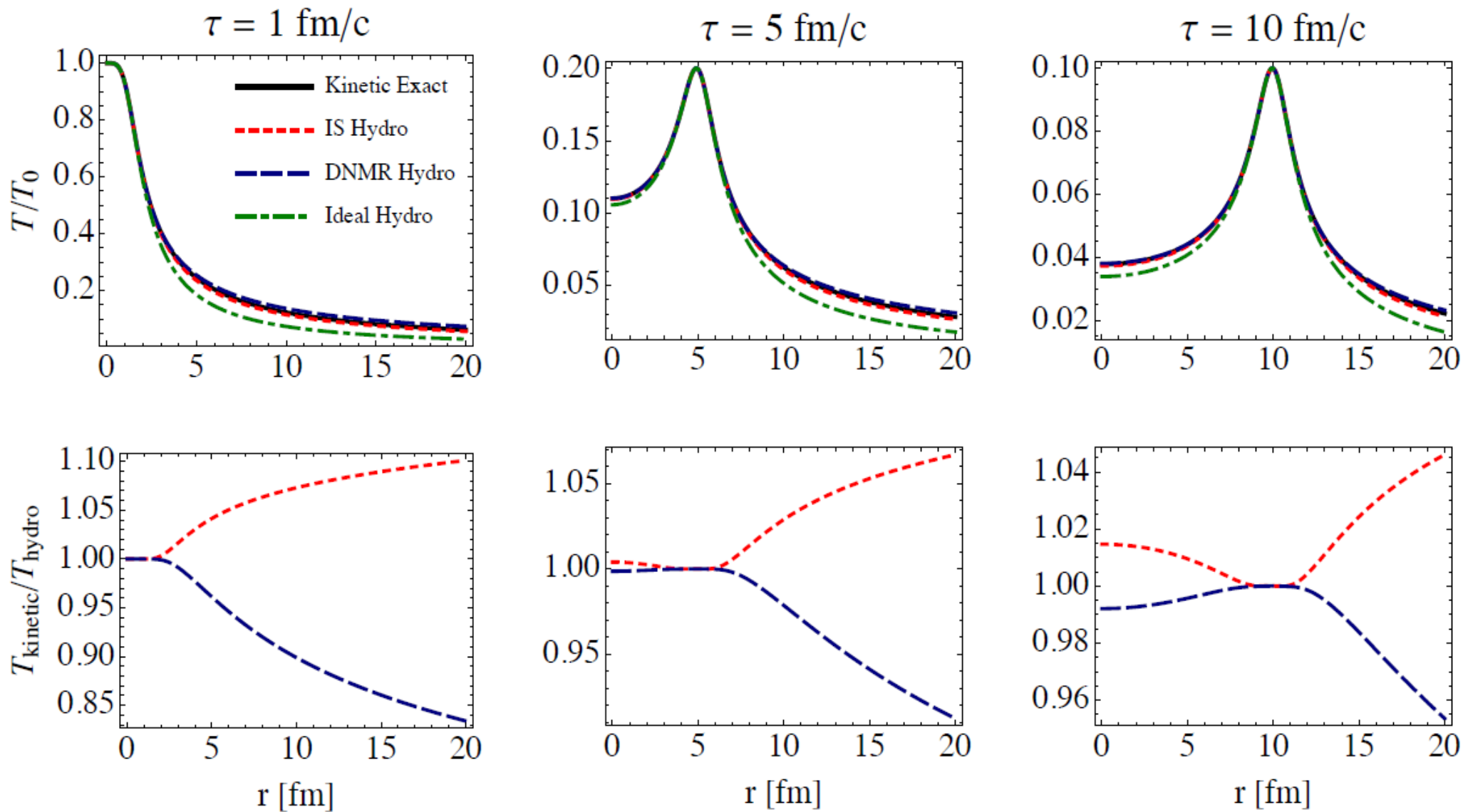


# Shear viscous tensor in Minkowski space



$$\bar{\pi}_\zeta^\zeta \equiv \pi_\zeta^\zeta / (\hat{T} \hat{S})$$

# Comparisons in Minkowski space: Temperature



$$\frac{\eta}{S} = \frac{1}{4\pi}$$

# Conclusions

- We find a new solution to the RTA Boltzmann equation undergoing simultaneous longitudinal and transverse expansion.
- We use this kinetic solution to test the validity and accuracy of different viscous hydrodynamical approaches.
- 2<sup>nd</sup> order viscous hydro provides a reasonable description when compared with the exact solution.
- This solution opens novel ways to test the accuracy of different hydro approaches

# Outlook and Perspectives

- Generalize this method for other conformal maps between Minkowski space and other curved spaces  
Denicol, Hatta, Martinez, Noronha and Xiao (forthcoming)
- Gubser exact solution for highly anisotropic systems (Nopoush, Ryblewski, Strickland)
- Study the evolution of the distribution function  
(M. Martinez and U. Heinz, forthcoming)
- Hopefully we can learn something about isotropization/thermalization problem by using symmetries...

**Backup slides**

# A short overview of the relativistic Boltzmann equation

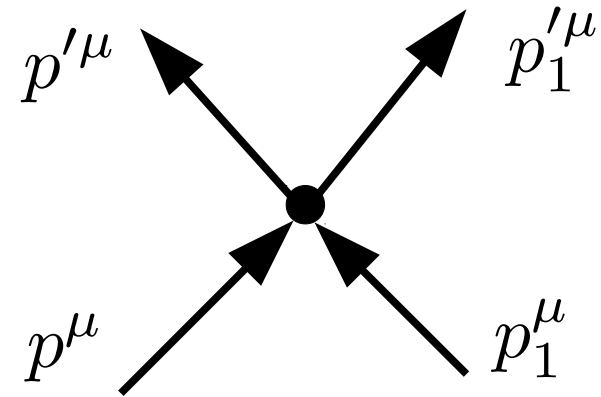
In a general curvilinear system and in the absence of external fields, the Boltzmann equation is written as

$$p^\mu \frac{\partial f}{\partial x^\mu} + \Gamma_{\mu i}^\lambda p_\lambda p^\mu \frac{\partial f}{\partial p_i} = \mathcal{C}[f]$$

The collisional kernel is given by

$$\mathcal{C}[f] = \int_{p', p'_1, p_1} \mathcal{W}(p, p_1 | p', p'_1) (f(x^\mu, p') f(x^\mu, p'_1) - f(x^\mu, p_1) f(x^\mu, p))$$

$\mathcal{W}(p, p_1 | p', p'_1)$  : transition rate



# Emergent conformal symmetry

A tensor  $(m,n)$  of canonical dimension  $\Delta$  transforms under a conformal transformation as

$$Q^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}(x) \rightarrow e^{(\Delta+m-n)\Omega(x)} Q^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}(x)$$

$\Omega$  is an arbitrary function.

The Boltzmann equation for massless particles is invariant under a conformal transformation (Baier et. al. JHEP 0804 (2008) 100)

$$p^\mu \partial_\mu f + \Gamma_{\mu i}^\lambda p_\lambda p^\mu \frac{\partial f}{\partial p_i} - \mathcal{C}[f] = 0$$



$$e^{2\Omega} \left( p^\mu \partial_\mu f + \Gamma_{\mu i}^\lambda p_\lambda p^\mu \frac{\partial f}{\partial p_i} - \mathcal{C}[f] \right) = 0$$

# For ideal hydro

From the E-M conservation law + ideal EOS + no viscous terms

$$\nabla_{\mu} T^{\mu\nu} = 0 \qquad p = \frac{\epsilon}{3} \qquad \eta = \zeta = 0$$

It follows this equation in the  $(\rho, \theta, \phi, \eta)$  coordinates

$$\frac{d}{d\rho} (\hat{\epsilon}^{3/4} \cosh^2 \rho) = 0$$

The solution is easy to find

$$\hat{\epsilon} = \hat{\epsilon}_0 (\cosh \rho)^{-8/3}$$

To go back to Minkowski space

$$\epsilon = \frac{\hat{\epsilon}}{\tau^4} = \frac{\hat{\epsilon}_0}{\tau^{4/3}} \frac{(2q)^{8/3}}{[1 + 2q^2(\tau^2 + r^2) + q^4(\tau^2 - r^2)^2]^{4/3}}$$

# Conformal Navier-Stokes solution

Let's preserve the conformal invariance of the theory

$$p = \frac{\epsilon}{3} \quad \eta = H_0 \epsilon^{3/4} \quad \zeta = 0$$

The temperature and the energy are related by

$$\hat{\epsilon} = \hat{T}^4$$

So from the EM conservation one obtains a solution for the temperature

$$\hat{T}(\rho) = \frac{\hat{T}_0}{(\cosh \rho)^{2/3}} \left[ 1 + \frac{H_0}{9\hat{T}_0} \sinh^3 \rho {}_2F_1 \left( \frac{3}{2}, \frac{7}{6}, \frac{5}{2}, -\sinh^2 \rho \right) \right]$$

These solutions predict **NEGATIVE** temperatures

# Conformal IS solution

In the de Sitter space the equations of motion are

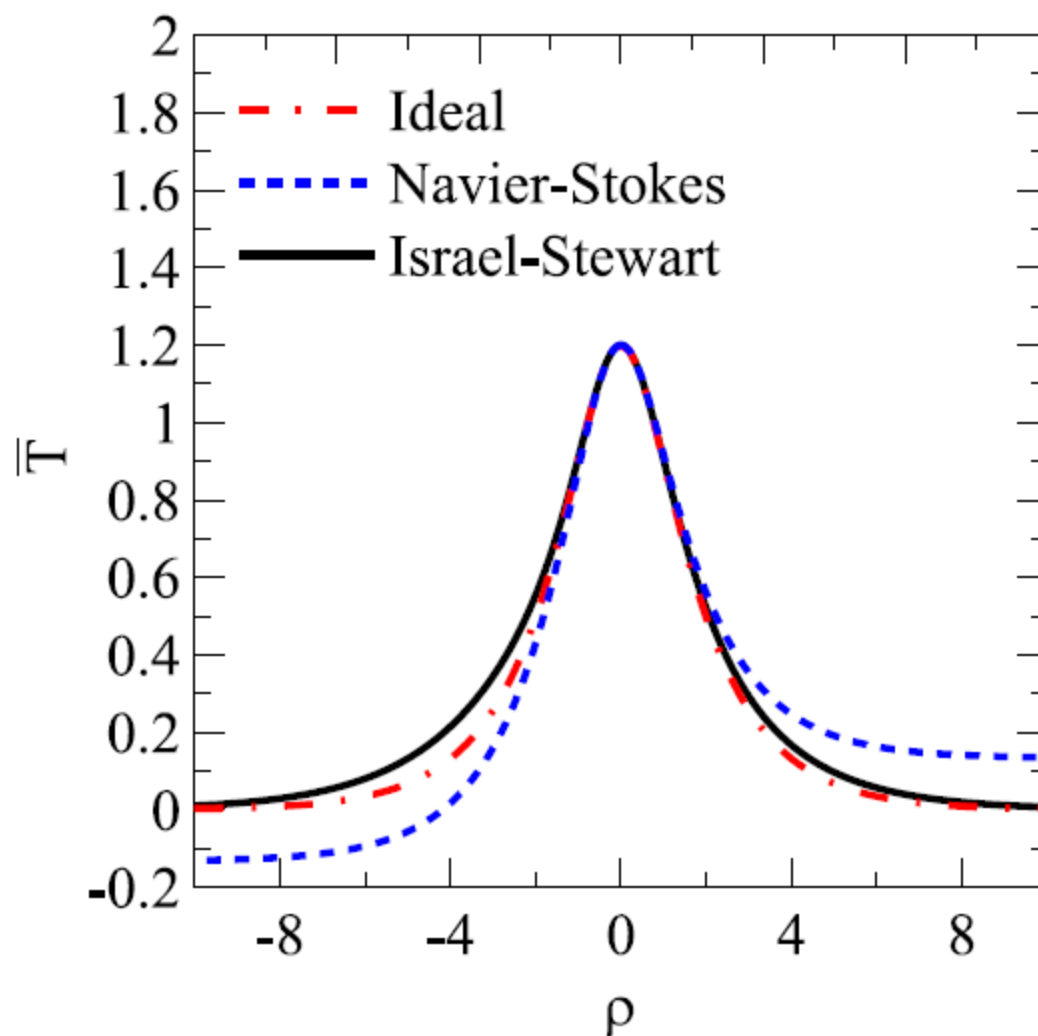
$$\frac{1}{\hat{T}} \frac{d\hat{T}}{d\rho} + \frac{2}{3} \tanh \rho = \frac{1}{3} \bar{\pi}_{\xi}^{\xi}(\rho) \tanh \rho ,$$
$$\frac{c}{\hat{T}} \frac{\eta}{s} \left[ \frac{d\bar{\pi}_{\xi}^{\xi}}{d\rho} + \frac{4}{3} \left( \bar{\pi}_{\xi}^{\xi} \right)^2 \tanh \rho \right] + \bar{\pi}_{\xi}^{\xi} = \frac{4}{3} \frac{\eta}{s \hat{T}} \tanh \rho ,$$

where in order to have conformal symmetry one assumes

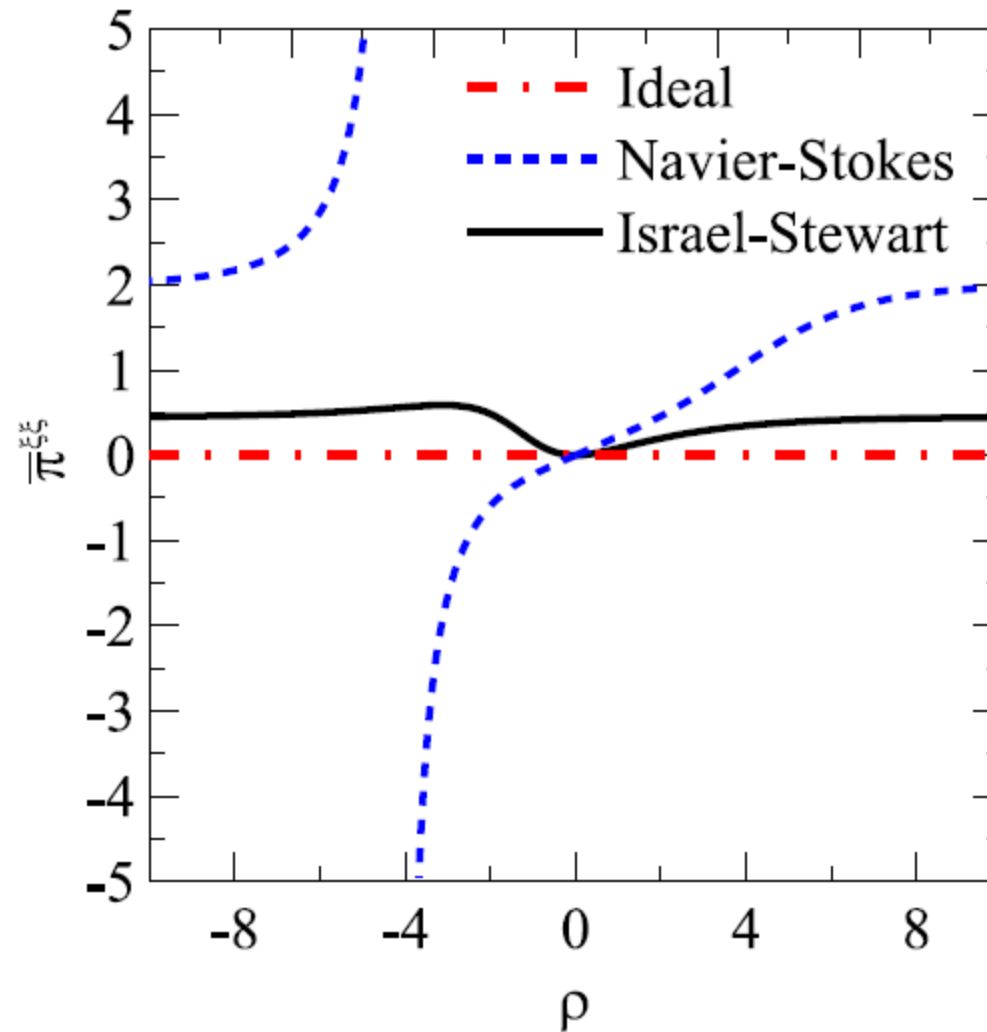
$$p = \frac{\epsilon}{3} \quad s \sim T^3 \quad \zeta = 0$$

$$\eta \sim s \quad \tau_R = c \eta / (T s)$$

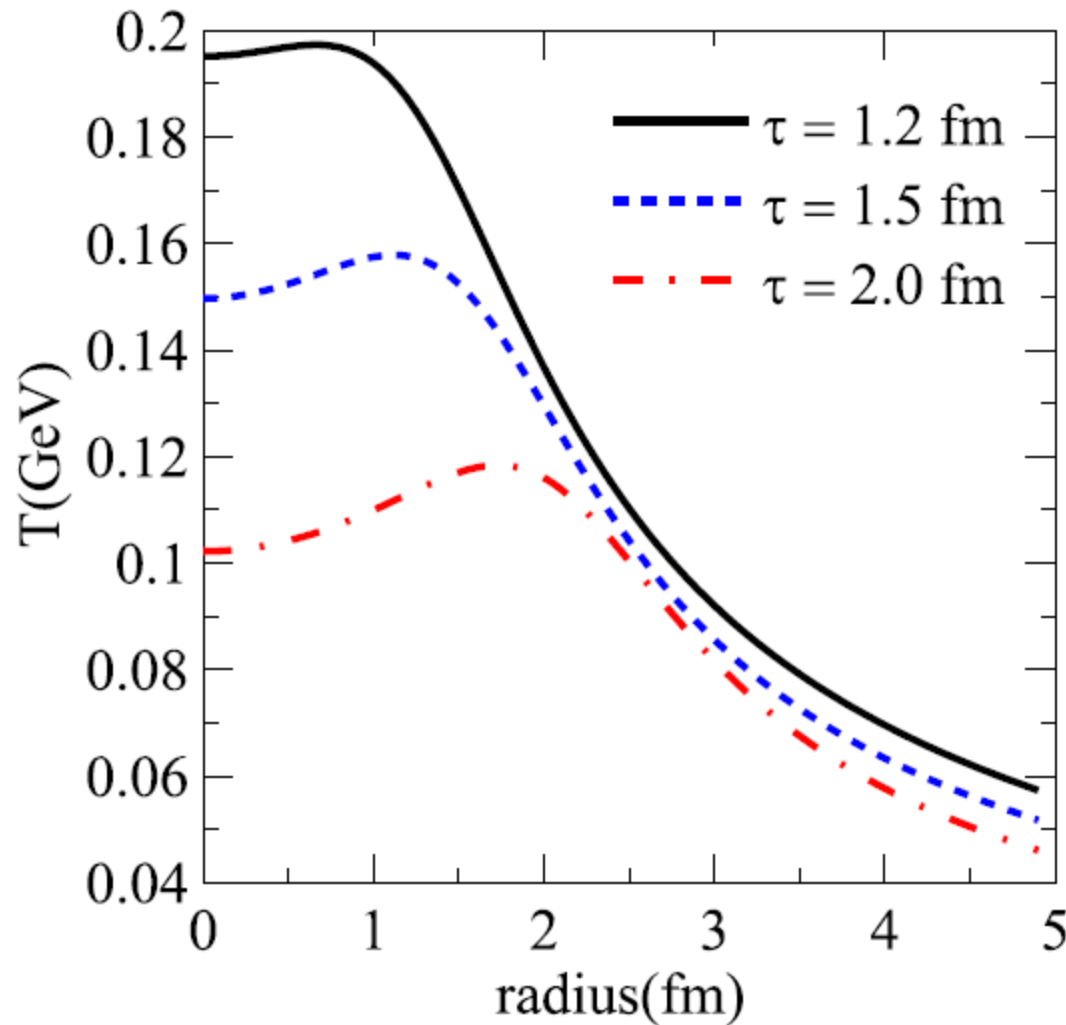
# Comparing Conformal Solutions



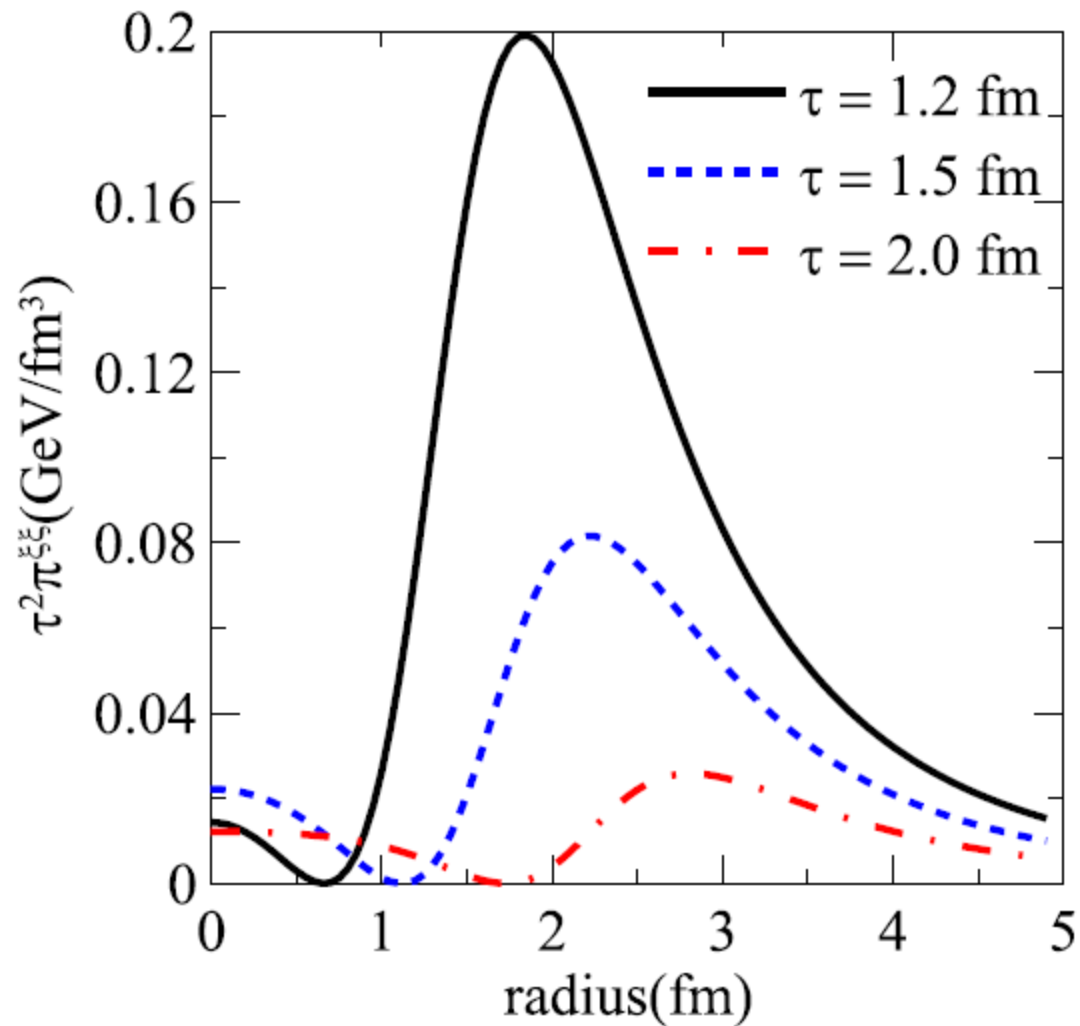
# Comparing Conformal Solutions



# Comparing Conformal Solutions



# Comparing Conformal Solutions



# Exact Conformal IS solution

These two equations are not solvable by analytical methods but in the cold plasma limit  $\eta/(s\hat{T}) \gg 1$

$$\bar{\pi}_{\xi}^{\xi}(\rho) = \sqrt{\frac{1}{c}} \tanh \left[ \sqrt{\frac{1}{c}} \left( \frac{4}{3} \ln \cosh \rho - \bar{\pi}_0 c \right) \right]$$

$$\hat{T}(\rho) = \hat{T}_1 \frac{\exp(c\bar{\pi}_0/2)}{(\cosh \rho)^{2/3}} \cosh^{1/4} \left[ \sqrt{\frac{1}{c}} \left( \frac{4}{3} \ln \cosh \rho - \bar{\pi}_0 c \right) \right]$$

# Gubser solution's for conformal hydrodynamics

The energy-momentum tensor of a conformal fluid

$$T^{\mu\nu} = u^\mu u^\nu (\varepsilon + \mathcal{P}) + g^{\mu\nu} \mathcal{P} + \pi^{\mu\nu}$$

From the energy-momentum conservation  $\nabla_\mu T^{\mu\nu} = 0$

$$\frac{d\hat{\varepsilon}(\rho)}{d\rho} + \frac{8}{3}\hat{\varepsilon} \tanh \rho - \hat{\pi}^{\eta\eta} \tanh \rho = 0$$

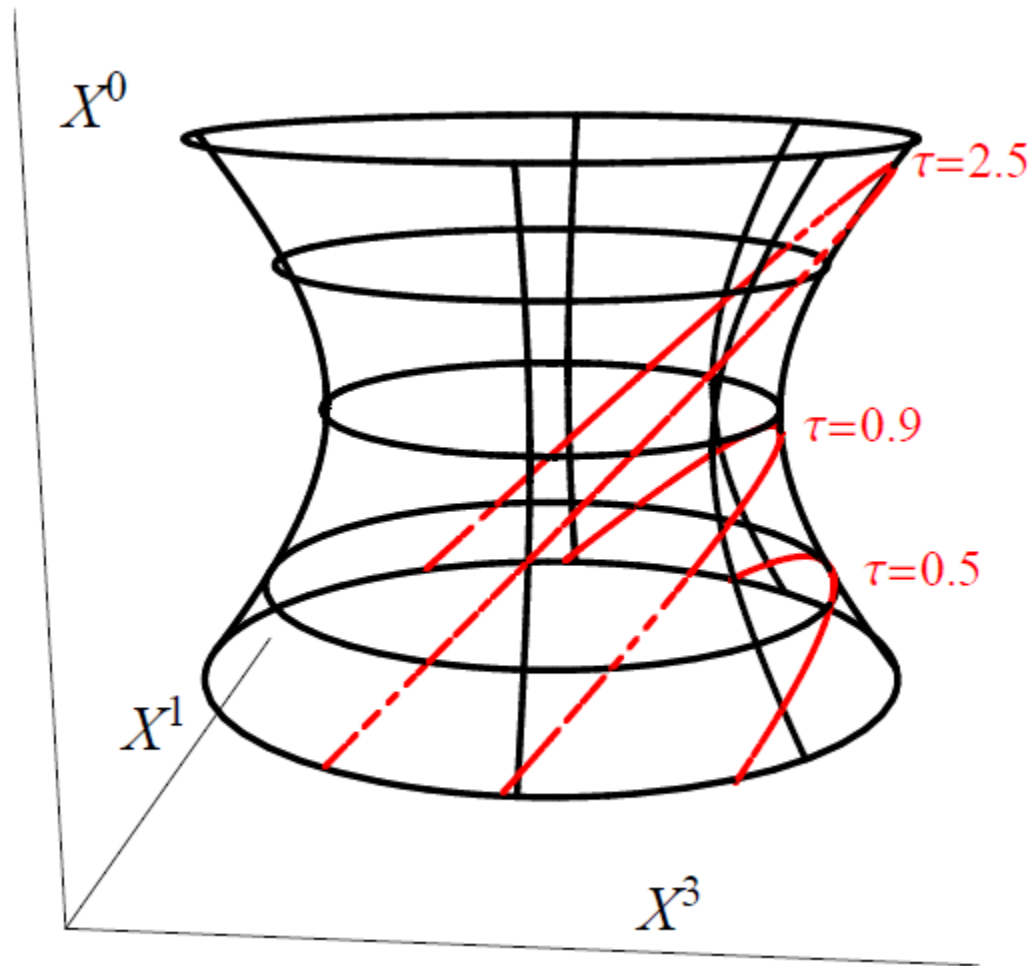
In IS theory the equation of motion of the shear viscous tensor  $\pi^{\mu\nu}$

$$\tau_{rel} \partial_\rho \hat{\pi}_{\langle\mu\nu\rangle} + \hat{\pi}_{\mu\nu} = -2\eta\sigma_{\mu\nu} - \frac{4}{3}\hat{\pi}_{\mu\nu}\theta$$

$$\theta = \partial_\mu u^\mu \quad \hat{\sigma}^{\mu\nu} = \hat{\Delta}_{\alpha\beta}^{\mu\nu} \partial^\alpha u^\beta$$

Ideal and NS solution (2010): Gubser, PRD82 (2010)085027, NPB846 (2011)469 Conformal IS theory (2013): Denicol et. al. arXiv:1308.0785

# A quick look to the de Sitter geometry



# Expansion models in $dS_3 \otimes R$

## Energy momentum conservation

$$\frac{1}{\hat{T}} \frac{d\hat{T}}{d\rho} + \frac{2}{3} \tanh \rho = \frac{1}{3} \bar{\pi}_\zeta^\zeta \tanh \rho$$

**Israel-Stewart (IS)**  $\longrightarrow \partial_\rho \bar{\pi}_\zeta^\zeta + \frac{\bar{\pi}_\zeta^\zeta}{\hat{\tau}_\pi} \tanh \rho + \frac{4}{3} (\bar{\pi}_\zeta^\zeta)^2 = \frac{4}{15} \tanh \rho$

**Denicol et. al. (DNMR)**  $\longrightarrow \partial_\rho \bar{\pi}_\zeta^\zeta + \frac{\bar{\pi}_\zeta^\zeta}{\hat{\tau}_\pi} \tanh \rho + \frac{4}{3} (\bar{\pi}_\zeta^\zeta)^2 = \frac{4}{15} \tanh \rho + \frac{10}{7} \bar{\pi}_\zeta^\zeta \tanh \rho$

**Ideal Hydro**  
 $\eta/s \rightarrow 0$   $\longrightarrow \hat{T}_{\text{ideal}}(\rho) = \frac{\hat{T}_0}{\cosh^{2/3}(\rho)} \quad \hat{\pi}^{\zeta\zeta} = 0$

**Free streaming**  
 $\eta/s \rightarrow \infty$   $\longrightarrow \hat{T}_{f.s.}(\rho) = \mathcal{H}^{1/4} \left( \frac{\cosh \rho_0}{\cosh \rho} \right) \hat{T}_0(\rho_0) \quad \hat{\pi}_{f.s.}^{\zeta\zeta}(\rho) = \mathcal{A} \left( \frac{\cosh \rho_0}{\cosh \rho} \right) \frac{\hat{T}_0^4}{\pi^2}$

$$\bar{\pi}_\zeta^\zeta \equiv \pi_\zeta^\zeta / (\hat{T} \hat{S}) \quad \hat{\tau}_\pi = 5\eta / (\hat{S} \hat{T})$$

# Weyl rescaling + Coordinate transformation

$$\rho = -\sinh^{-1} \left( \frac{1 - q^2 \tau^2 + q^2 r^2}{2q\tau} \right)$$

$$\theta = \tanh^{-1} \left( \frac{2qr}{1 + q^2 \tau^2 - q^2 r^2} \right)$$

$$\rho \in [-\infty, \infty]$$

$$0 < \theta < 2\pi$$

$\rho$  is the affine parameter  
(e.g. “time”)

